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Light Velocity in Nonrelativistic Quantum Mechanics on a Circle

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A single-valued Schrödinger wave function on a circle is discussed in the framework of the path integral. It is clarified how the light velocity comes into play and how terms corresponding to superluminal transmission of signals are suppressed.

1. INTRODUCTION

In this paper we investigate Schrödinger wave functions on a circle, which is the simplest case of multiply connected spaces. We use the Feynman path integral approach to quantum mechanics, although our result is independent of the formalism. It is known that single-valued wave functions for multiply connected spaces are obtained when we add up the contributions from the paths corresponding to all the homotopy classes with an equal weight (Schulman, 1968, 1971; Berry, 1980; Yabuki, 1986). For the one-dimensional Lagrangian

$$L = \tfrac{1}{2}m\dot{x}^2 \tag{1}$$

the ordinary Feynman kernel for the transition $(t_0, x_0) \to (t, x)$ is given by

$$K(x, t; x_0, t_0) = \left(\frac{2\pi i\hbar(t - t_0)}{m}\right)^{-1/2} \exp\frac{im(x - x_0)^2}{2\hbar t} \tag{2}$$

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When the particle moves on a circle with radius a, we have the following kernel:

$$K_c(x, t; x_0, t_0) = \sum_{n=-\infty}^{\infty} \left(\frac{2\pi i\hbar(t - t_0)}{m}\right)^{-1/2} \exp \frac{im(x + 2\pi an - x_0)^2}{2\hbar t} \tag{3}$$

where $-\pi a \leq x_0 < \pi a$, $-\pi a \leq x < \pi a$. The integer n denotes the homotopy class corresponding to the path winding the circle n times. The above kernel $K_c(x, t; x_0, t_0)$ gives a single-valued wave function $\psi(x, t)$ for an initial single-valued $\psi(x_0, t_0)$.

When the initial wave function $\psi(x_0, t_0 = 0)$ is the Dirac delta function $\delta(x_0)$, then $\psi(x, t) = K_c(x, t; 0, 0)$ for $t > 0$. In this case the probability amplitude for finding the particle at x (for $0 < t < \pi a/c$, say) receives nonnegligible contributions even from large n's, since each term in the sum in equation (3) is of equal magnitude. Here c denotes the light velocity. (In a space whose dimension is greater than or equal to 2, contributions coming from nontrivial homotopy classes usually correspond to big deviations from classical paths and result in negligibly small values due to cancellations among rapidly oscillating factors of nearby paths.) In our case, nonnegligible contributions from terms with $n \neq 0$ (for $0 < t < \pi a/c$) mean the transmission of signals faster than light.

Here, however, we are faced with a pathological situation (Schulman, 1981). This can be seen by rewriting $K_c(x, t; 0, 0)$ as follows:

$$\begin{aligned} K_c(x, t; 0, 0) &= \left(\frac{2\pi i\hbar t}{m}\right)^{-1/2} \exp\left(\frac{imx^2}{2\hbar t}\right) \sum_{n=-\infty}^{\infty} \exp\left(\frac{i2\pi^2 a^2 mn^2}{\hbar t}\right) \exp\left(\frac{i2\pi manx}{\hbar t}\right) \\ &= \left(\frac{2\pi i\hbar t}{m}\right)^{-1/2} \exp\left(\frac{imx^2}{2\hbar t}\right) \vartheta_3(\nu, \tau) \end{aligned} \tag{4}$$

where $\vartheta_3(\nu, \tau)$ is the Jacobi theta function (Whittaker and Watson, 1927),

$$\vartheta_3(\nu, \tau) = \sum_{n=-\infty}^{\infty} \exp(i\pi\tau n^2) \exp(i2\pi n\nu) \tag{5}$$

with

$$\nu = \frac{amx}{\hbar t}, \qquad \tau = \frac{2\pi ma^2}{\hbar t} \tag{6}$$

It is known that $\vartheta_3(\nu, \tau)$ is analytic for $\mathrm{Im}(\tau) > 0$ and $|\nu| < \infty$ with quasiperiodicity in ν with periods 1 and τ, and that it is zero at $(1 + \tau)/2$. From these facts it follows that zeros of $\vartheta_3(\nu, \tau)$ are dense on the real axis of ν if τ is real and irrational.

Therefore, we cannot draw any physical conclusion from the expression (4) as it stands.

2. SUPPRESSION OF SUPERLUMINAL TRANSMISSION OF SIGNALS

In this section we would like to clarify the above situation, starting from the following 'realistic' choice for our initial state $\psi(x_0, 0)$, $-\pi a \leq x_0 < \pi a$:

$$\psi(x_0, 0) = \frac{1}{\sqrt{2\pi s}} \exp\left[-\frac{1}{2}\left(\frac{x_0}{s}\right)^2\right] \tag{7}$$

Then the wave function $\psi(x, t)$, $t > 0$, is obtained as follows:

$$\begin{aligned}
\psi(x, t) &= \int_{-\pi a}^{\pi a} K_c(x, t; x_0, 0)\psi(x_0, 0)\, dx_0 \\
&\simeq \sum_{n=-\infty}^{\infty} \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} \exp\left(-\frac{x_0^2}{2s^2} + i\,\frac{m(x_0 - x - 2\pi a n)^2}{2\hbar t}\right) dx_0 \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{s^2 + iw^2}} \exp\left(i\,\frac{(x + 2\pi a n)^2}{2(w^2 - is^2)}\right) \\
&= \frac{1}{\sqrt{2\pi}\sqrt{s^2 + iw^2}} \exp\left(\frac{ix^2}{2(w^2 - is^2)}\right)\vartheta_3(v, \tau)
\end{aligned} \tag{8}$$

where we defined

$$w = \left(\frac{\hbar t}{m}\right)^{1/2}, \qquad v = \frac{ax}{w^2 - is^2}, \qquad \tau = \frac{2\pi a^2}{w^2 - is^2} \tag{9}$$

The appearance of the positive imaginary part in τ,

$$\mathrm{Im}(\tau) = 2\pi a^2 s^2/(w^4 + s^4) \tag{10}$$

means exponential damping in the series expansion (5) of the Jacobi theta function.

Let us first consider the case where the ratio

$$\left(\frac{s}{w}\right)^2 = \frac{s}{\lambda}\,\frac{s}{ct} \tag{11}$$

is much smaller than 1, where we have put the Compton wavelength of the particle to be λ,

$$\lambda = \frac{\hbar}{mc} \tag{12}$$

(For example, if the width of the wave packet of the initial wave function of an electron $s = 10^{-3}$ cm and $t = 10^{-2}$ sec, we have $s/w \simeq 10^{-2}$.) In this case we have a damping factor $\exp - (2\pi asn/w^2)^2/2$ in the series expansion of $\vartheta_3(\nu, \tau)$. Since the exponent of this factor is minus one-half of the square of the ratio

$$\frac{2\pi asn}{w^2} = \frac{s}{\lambda}\frac{2\pi an}{ct} \tag{13}$$

we find that the series in $\vartheta_3(\nu, \tau)$ receives contributions only from terms with n such that $2\pi a|n| < ct$, since the ratio s/λ cannot be made less than 1 in realistic situations. The same conclusion may be drawn from the second exponential damping factor in (5), $\exp[-(s/w)^2(x/\lambda)(2\pi an/ct)]$, if $(s/w)^2(x/\lambda) > 1$. This result means that the terms corresponding to the superluminal transmission of signals ($2\pi a|n| > ct$) are exponentially suppressed. The velocity of light appears naturally in nonrelativistic quantum mechanics through the Compton wavelength of the particle.

On the other hand, if the ratio s/w is bigger than 1, then the exponential damping factor is roughly $\exp[-(2\pi an/s)^2/2] \times \exp[-(x/s)(2\pi an/s)]$. In this case, the term with $n = 0$ dominates in $\vartheta_3(\nu, \tau)$, since s, the initial spread of the wave function, must be much smaller than $2\pi a$ in order to give meaning to our problem.

Up to now, we considered only the Jacobi theta function $\vartheta_3(\nu, \tau)$ in equation (8). There is another factor $\exp[(i/2)(x^2/(w^2 - is^2))]$ in front of $\vartheta_3(\nu, \tau)$ in equation (8). The analysis of this factor with respect to its behavior for large $|x|$ gives a result consistent with the above conclusion.

3. DISCUSSION

In our one-dimensional case investigated above, the sum over the paths corresponding to a nontrivial homotopy class n ($\neq 0$) gives the contribution

$$\left(\frac{2\pi i\hbar(t - t_0)}{m}\right)^{-1/2} \exp\frac{im(x + 2\pi an - x_0)^2}{2\hbar t} \tag{14}$$

to the *integration kernel* $K_c(x, t; x_0, t_0)$. The magnitude of this term is the same as that with $n = 0$. When we look back at the argument of the preceding section, we find that the suppression of the superluminal propagation of the particle has resulted from the cancellation among contributions from the paths that differ slightly in their respective *starting points*. In fact, we integrated over the contributions from these paths by putting a Gaussian weight in the initial wave function. See the second line of equation (8). (It would be easy to convince oneself that the superposition of terms of different final points gives another suppression of superluminal paths.)

Therefore, this mechanism for the suppression of nonclassical, superluminal paths is essentially the same as in the case of higher dimensional spaces, namely the cancellation among nearby paths. The difference is that, in the case of higher dimensions, nonclassical paths sum up to give, ordinarily, contributions to the integration *kernel* which are very much smaller in magnitude than the main term corresponding to the classical path, that is, we have the suppression *before* the superposition of terms of different initial points.

In conclusion, concerning actual situations, the light velocity c comes into play from the inevitable limitation of the size of localization of the particle, that is, from the Compton wavelength $\hbar/mc$ of the particle in question, and we have the suppression of contributions from paths corresponding to superluminal travel. In spite of the fact that quantum mechanics is a nonrelativistic theory, its wave nature manages to cancel out communications faster than light.

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